CALCULATION OF TEMPERATURE AND MOISTURE FIELDS WITH A PULSATING INTERNAL HEAT SOURCE

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A solution is obtained to the system of differential equations of heat and mass transfer with a pulsating internal heat source and with boundary conditions of the third kind.

High-frequency pulse heating of materials is often more effective than desiccation by continuous heating [1, 7, 10]. For this reason, an analysis of that process would be of theoretical and practical value.

The one-dimensional problem of heat and moisture transfer through a material containing a positive internal heat source and constrained by boundary conditions of the third kind can be formulated as follows [3]:

$$\left.\begin{aligned} \frac{\partial T(X,\ \mathrm{Fo})}{\partial \mathrm{Fo}} &= \frac{\partial^2 T(X,\ \mathrm{Fo})}{\partial X^2} - \varepsilon\,\mathrm{Ko}\,\frac{\partial \Theta(X,\ \mathrm{Fo})}{\partial \mathrm{Fo}} - \mathrm{Po}(X,\ \mathrm{Fo}),\\ \frac{\partial \Theta(X,\ \mathrm{Fo})}{\partial \mathrm{Fo}} &= \mathrm{Lu}\,\frac{\partial^2 \Theta(X,\ \mathrm{Fo})}{\partial X^2} - \mathrm{Pn}\,\mathrm{Lu}\,\frac{\partial^2 T(X,\ \mathrm{Fo})}{\partial X^2}, \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} -\frac{\partial T(1,\ \mathrm{Fo})}{\partial X} + \mathrm{Bi}_q T(1,\ \mathrm{Fo}) + (1-\varepsilon)\,\mathrm{Ko}\,\mathrm{Lu}\,\mathrm{Bi}_m \Theta(1,\ \mathrm{Fo}) &= 0,\\ -\frac{\partial \Theta(1,\ \mathrm{Fo})}{\partial X} + \mathrm{Pn}\,\frac{\partial T(1,\ \mathrm{Fo})}{\partial X} + \mathrm{Bi}_m \Theta(1,\ \mathrm{Fo}) &= 0,\\ \frac{\partial T(0,\ \mathrm{Fo})}{\partial X} = \frac{\partial \Theta(0,\ \mathrm{Fo})}{\partial X} &= 0,\\ T(X,\ 0) = T_0(X),\ \ \Theta(X,\ 0) &= \Theta_0(X), \end{aligned}\right\} \tag{2}$$

where

$$T(X,\ \mathrm{Fo}) = \frac{t_a - t(x,\ \tau)}{t_a - t_s},\qquad \Theta(X,\ \mathrm{Fo}) = \frac{\vartheta(x,\ \tau) - \vartheta_e}{\vartheta_s - \vartheta_e},$$
$$-1 < X < +1,\quad 0 < \mathrm{Fo} < \infty. \tag{3}$$

All critical numbers here are assumed constant, except the Pomerantsev number and the Fourier number.

By the method shown in [5, 9], this problem is reduced to two independent equations of the heat conduction kind with the following initial and boundary conditions:

$$\frac{\partial Z_i(X,\ \mathrm{Fo})}{\partial \mathrm{Fo}} = \frac{1}{\nu_i^2}\cdot\frac{\partial^2 Z_i(X,\ \mathrm{Fo})}{\partial X^2} - p_i\,\mathrm{Po}(X,\ \mathrm{Fo}), \tag{4}$$

$$\frac{\partial Z_i(1,\ \mathrm{Fo})}{\partial X} + \mu_i^2 Z_i(1,\ \mathrm{Fo}) = 0,\qquad \partial Z_i\,\frac{(0,\ \mathrm{Fo})}{\partial X} = 0, \tag{5}$$

$$Z_i(X,\ 0) = Z_{0i}(X),\quad i = 1;\ 2, \tag{6}$$

where the Z_i variables represent transfer potentials, as linear combinations of T and Θ. Inside the material

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$$Z_i(X,\ \mathrm{Fo}) = p_i T(X,\ \mathrm{Fo}) + q_i \Theta(X,\ \mathrm{Fo}), \tag{7}$$

and on its surface

$$Z_i(1,\ \mathrm{Fo}) = m_i T(1,\ \mathrm{Fo}) + n_i \Theta(1,\ \mathrm{Fo}). \tag{8}$$

The constant coefficients p, q, m, and n are defined as follows:

$$p_i = \left(\frac{\mathrm{Pn}}{\nu_2^2 - 1}\right)^{i-1}, \quad q_i = \left(\frac{\mathrm{Pn}}{\nu_1^2 - 1}\right)^{i-2},$$

$$m_i = \left(\frac{\mathrm{Pn\,Bi}_q}{\mu_2^2 - \mathrm{Bi}_q}\right)^{i-1}, \quad n_i = \left(\frac{\mathrm{Pn\,Bi}_q}{\mu_1^2 - \mathrm{Bi}_q}\right)^{i-2},$$

where

$$\nu_i^2 = \frac{1}{2}\left(\Phi_1 + (-1)^i \sqrt{\Phi_1^2 - \frac{4}{\mathrm{Lu}}}\right),$$

$$\mu_i^2 = \frac{1}{2}\left(\Phi_2 + (-1)^i \sqrt{\Phi_2^2 - 4\,\mathrm{Bi}_q \mathrm{Bi}_m}\right),$$

$$\Phi_1 = 1 + \varepsilon\,\mathrm{Ko\,Pn} + \frac{1}{\mathrm{Lu}}, \quad \Phi_2 = \mathrm{Bi}_q + \mathrm{Bi}_m[1 + (1-\varepsilon)\,\mathrm{Ko\,Pn\,Lu}],$$

ν_i^2 and μ_i^2 are the roots of the characteristic equations reducing system (1) with the boundary condition (2) to Eqs. (4) with the boundary condition (5).

A simultaneous transformation of system (1) into Eqs. (4) and boundary conditions (2) of the third kind into boundary conditions (5) is possible, if the characteristic equations of the system and of the boundary conditions have common roots [5]. In this case an inversion to transfer potentials T and Θ according to formula (7) is possible, after problem (4)-(6) has been solved.

We proceed on the assumption that the internal heat source can be expressed as $\mathrm{Po}(X,\ \mathrm{Fo}) = \mathrm{Po} f_1(X) f_2(\mathrm{Fo})$.

With the aid of Fourier and Laplace integral transformations, the solution to problem (4)-(6) has been obtained for two special cases with the same initial potential distribution $T_0(X) = 1 + (1 - X^2)U$ and $\Theta_0(X) = 1 + (1 + X^2)V$ but with different pulse functions of the internal heat source.

Case 1.

$$\mathrm{Po}(X,\ \mathrm{Fo}) = \mathrm{Po}(1 - WX^2)\left(1 - \cos\frac{2\pi}{\mathrm{Fo_s}}\mathrm{Fo}\right).$$

Problem (4)-(6) was solved by applying the Laplace transformation with respect to both variables X and Fo [2]. A subsequent inverse Laplace transformation yields

$$Z_i(X,\ \mathrm{Fo}) = \sum_{n=1}^{\infty} A_{ni} B_{ni} \exp(-k_{ni}^2 \mathrm{Fo}) \cos \nu_i k_{ni} X$$
$$- p_i \mathrm{Po}\left\{1 + \frac{1}{2}\nu_i^2(1 - X^2) - \frac{1}{3}W\left[1 + \frac{1}{4}\nu_i^2(1 - X^4)\right]\right\}$$
$$+ p_i \mathrm{Po} \sum_{j=1}^{2}\sum_{n=1}^{\infty} A_{ni} C_{nij} \exp(-k_{ni}^2 \mathrm{Fo}) \cos \nu_i k_{ni} X, \tag{9}$$

where

$$A_{ni} = \frac{2 \sin \nu_i k_{ni}}{\nu_i k_{ni} + \sin \nu_i k_{ni} \cos \nu_i k_{ni}},$$

$$B_{ni} = p_i + q_i - \frac{2(k_{ni}^2 - 1)}{\nu_i k_{ni}}(p_i U + q_i V), \quad C_{ni1} = \frac{D_{ni}}{k_{ni}^2},$$

$$C_{ni2} = \frac{D_{ni}}{\left(\frac{2\pi}{\mathrm{Fo_s}}\right)^2 + k_{ni}^4}\left[\frac{2\pi}{\mathrm{Fo_s}}\sin\frac{2\pi}{\mathrm{Fo_s}}\mathrm{Fo} + k_{ni}^2\cos\frac{2\pi}{\mathrm{Fo_s}}\mathrm{Fo}\right.$$

$$\left. - k_{ni}^2\exp(-k_{ni}^2\mathrm{Fo})\right]\exp(k_{ni}^2\mathrm{Fo}),$$

$$D_{ni} = 1 - W\left[1 + \frac{2(k_{ni}^2 - 1)}{\nu_i^2 k_{ni}^2}\right],$$

$$U = \frac{t_s - t_c}{t_a - t_s}, \quad V = \frac{\vartheta_c - \vartheta_s}{\vartheta_s - \vartheta_e}, \quad W = \frac{\mathrm{Po} - \mathrm{Po_s}}{\mathrm{Po}},$$

k_{ni} are the roots of the characteristic equation $\cot_i k = k/\nu_i$, and Po is the value of the Pomerantsev number at X = Fo = 0.

When the potential and the heat source are distributed uniformly, U = V = W = 0 and the pulse function vanishes ($C_{ni2} = 0$), then the solution becomes the well-known solution [4] to the equation of heat conduction with a constant internal heat source.

When moist materials are heated dielectrically, then the variation of the internal heat source with time is more properly represented by a Π-pulse function. Since the dielectric losses in the material decrease with the removal of moisture, moreover, the value of this function decreases with time.

Case 2.

$$\mathrm{Po}(X, \mathrm{Fo}) = \mathrm{Po}(1 - WX^2) f(\mathrm{Fo}) \exp(-\mathrm{Pd\,Fo}),$$

where f(Fo) is a Π-pulse function.

Unlike in the preceding case, here we apply the Fourier cosine transformation [6, 8] with respect to variable X. With respect to variable Fo, as before, we apply the Laplace transformation. Such a combination of integral transformations yields the solution to the problem in a simpler form than would be obtained by applying the Laplace integral transformation with respect to both variables. A Fourier cosine transformation in the preceding case yields the same solution (9).

After necessary operations, we have

$$Z_i(X, \mathrm{Fo}) = \sum_{n=1}^{\infty} A_{ni}B_{ni}\exp(-k_{ni}^2\mathrm{Fo})\cos\nu_i k_{ni}X$$

$$+ \frac{1}{2}\eta_\tau p_i \mathrm{Po}\frac{1}{\mathrm{Pd}}\exp(-\mathrm{Pd\,Fo})\left[E_i Q_i(X) + W\left(1 - X^2 + \frac{2}{\nu_i^2}\right)\right]$$

$$+ p_i\mathrm{Po}\sum_{j=1}^{2}\sum_{n=1}^{\infty} A_{ni}C'_{nij}\exp(-k_{ni}^2\mathrm{Fo})\cos\nu_i k_{ni}X, \qquad (10)$$

where

$$0 < \eta_\tau = \frac{\mathrm{Fo}_q}{\mathrm{Fo_s}} < 1, \quad E_i = 1 - W\left[1 + \frac{2(\mathrm{Pd} - 1)}{\nu_i^2\mathrm{Pd}}\right],$$

$$Q_i(X) = 1 - \frac{\cos\nu_i\sqrt{\mathrm{Pd}}X}{\cos\nu_i\sqrt{\mathrm{Pd}} - \nu_i^{-1}\sqrt{\mathrm{Pd}}\sin\nu_i\sqrt{\mathrm{Pd}}}, \qquad C'_{ni1} = \frac{B'_{ni}D_{ni}}{k_{ni}^2 - \mathrm{Pd}},$$

$$C'_{ni2} = -\exp(-\mathrm{Pd\,Fo})\frac{2}{\pi}\sum_{m=1}^{\infty} D_{mni}D_{ni}\sin\left[\frac{m\pi}{\mathrm{Fo_s}}(2\mathrm{Fo}\right.$$

$$\left. - \eta_\tau\mathrm{Fo_s}) + \psi_{mni}\right]\exp(k_{ni}^2\mathrm{Fo}),$$

$$B'_{ni} = \frac{\exp[\eta_\tau\mathrm{Fo_s}(k_{ni}^2 - \mathrm{Pd})] - 1}{\exp[\mathrm{Fo_s}(k_{ni}^2 - \mathrm{Pd})] - 1},$$

$$D_{mni} = \frac{\sin m\pi\eta_\tau}{m\sqrt{\left(\frac{2m\pi}{\mathrm{Fo_s}}\right)^2 + (k_{ni}^2 - \mathrm{Pd})^2}},$$

$$\psi_{mni} = \operatorname{arctg}\frac{\mathrm{Fo_s}(k_{ni}^2 - \mathrm{Pd})}{2m\pi}.$$

If U = V = W = 0 and the internal heat source is continuous in time (f(Fo) = 1), then solution (10) coincides with the solution to the problem of heat conduction with an exponential internal heat source [4].

NOTATION

$\mathrm{Fo_s}$ is the dimensionless period of the pulse function;
$\mathrm{Fo_q}$ is the dimensionless heating time;
U, V, W are the nonuniformity of the heat-transfer potential, the mass transfer potential, and the internal heat source distribution respectively.

Subscripts

s denotes surface;
c denotes center;
e denotes equilibrium;
a denotes ambient medium;
remaining notation as in [3].

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